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Relativistic Thomas-Fermi formulations with thermal effects for an atom within and without a very strong magnetic field

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In this paper, unified relativistic formulations with thermal effects of the Thomas-Fermi model for an atom within and without a very strong magnetic field are considered. The general formulations are then used to discuss the combined relativistic and thermal effects on some atomic properties.

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I. INTRODUCTION

The Thomas-Fermi model (Thomas [1], Fermi [2]) provides a heuristic semiclassical statistical method to describe the electrostatic potential and the electron distribution around the atomic nucleus, in which the electrons are considered to form an ideal gas obeying the Fermi-Dirac statistics. Relativistic corrections to this formulation become appreciable as the atomic number increases, because for very heavy atoms the electron velocities in the vicinity of the nucleus become relativistic. Relativistic corrections were developed by Vallarta and Rosen [3] by replacing the nonrelativistic electron kinetic-energy term in the Thomas-Fermi formulation by its relativistic counterpart. On the other hand, the thermal effects on the Thomas-Fermi formulation were considered by Marshak and Bethe [4] and Feynman, Metropolis, and Teller [5], among others (the thermal effects are as important as the relativistic effects if the temperatures considered are of the order of $10^7\kappa$ or more).

The study of atoms in very strong magnetic fields, on the other hand, is of interest in connection with the emission of ions and electrons from pulsars. (The pulsar in the Crab nebula has a magnetic field of the order of 6×10^{12} G.) In recognition of this astrophysical interest, Kadomtsev [6] formulated a modified Thomas-Fermi model to describe the ground state of a heavy atom in a very strong magnetic field. Relativistic corrections to this formulation were developed by Hill, Grout, and March [7] and Shivamoggi and Mulser [8], while the thermal effects were considered by Shivamoggi and Schram [9].

In this paper, we will consider unified relativistic formulations with thermal effects of the Thomas-Fermi model for an atom within and without a very strong magnetic field. (For the case of an atom without a magnetic field, such attempts had been made previously by Cox and Guilli [10] and Koester and Chanmugam [11] in connection with degenerate matter in white dwarfs, but those considerations were mainly numerical and did not go into atomic properties.) We will then use those general formulations to discuss the combined relativistic and thermal effects on some atomic properties.

II. GENERALIZED THOMAS-FERMI THEORY FOR AN ATOM

A. Relativistic Thomas-Fermi equation with thermal effects

Consider a heavy atom with an atomic number Z and let $\varphi(\mathbf{r})$ be the Coulomb potential due to the nucleus and the distribution of electronic charge around it. If $p_F(\mathbf{r})$ is the Fermi momentum, the number density of the electrons in the ground state (taken to be spherically symmetric) of the atom is given by

$$n(r)=\frac{1}{\pi^2\hbar^3}\int_0^\infty \frac{p^2dp}{\exp\left[\dfrac{\sqrt{p^2c^2+m^2c^4}-mc^2-e(\varphi-\varphi_0)}{KT}\right]+1}, \tag{1}$$

where

$$\varphi_0\equiv-\frac{\mu}{e}$$

and μ is the chemical potential of this electronic system.

The evaluation of the integral in (1) (without the Coulomb potential φ) has been given by Chandrasekhar [13] in the astrophysical context. We give below, nevertheless, the main steps in this calculation, since they are needed in the evaluation of similar integrals in the rest of the paper.

Making Juttner's [12] transformation

$$\frac{p}{mc}=\sinh\theta, \tag{2}$$

(1) becomes

$$n(r)=\frac{m^3c^3}{\pi^2\hbar^3}\int_0^\infty \frac{\sinh^2\theta\cosh\theta\,d\theta}{\frac{1}{\Lambda}e^{\beta mc^2\cosh\theta}+1}, \tag{3}$$

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where

$$\beta=\frac{1}{KT}, \quad \frac{1}{\Lambda}\equiv e^{-\beta[\mu+mc^2+e\varphi(r)]} .$$

Let us put (Chandrasekhar [13])

$$u=\beta mc^2 \cosh\theta , \tag{4}$$

so that (3) can be written as

$$n(r)=\frac{m^3c^3}{\pi^2\hbar^3}\frac{1}{\beta mc^2}\int_0^\infty \frac{d\Psi(u)/du}{\frac{1}{\Lambda}e^u+1}du , \tag{5}$$

where

$$\frac{d\Psi(u)}{du}=\tfrac{1}{2}\sinh 2\theta ,$$

Now the integral in (5) can be evaluated by using Sommerfeld's [14] lemma,

$$\int_0^\infty \frac{d\Psi(u)/du}{\frac{1}{\Lambda}e^u+1}du=\Psi(u_0)+\frac{\pi^2}{6}\Psi''(u_0)+\cdots , \tag{6}$$

which, in the present context, corresponds to the weak thermal-effect limit. Here,

$$u_0\equiv \ln\Lambda=\beta[\mu+mc^2+e\varphi(r)]\equiv\beta mc^2\cosh\theta_0 .$$

Introducing

$$q=\sinh\theta_0=\left[\frac{(\mu+mc^2+e\varphi)^2}{m^2c^4}-1\right]^{1/2}=\frac{p_F}{mc} , \tag{7}$$

where $p_F(\mathbf{r})$ is the Fermi momentum, we obtain

$$\Psi(u_0)=\frac{\beta mc^2}{3}q^3, \quad \Psi''(u_0)=\frac{1}{\beta mc^2}\frac{2q^2+1}{q} . \tag{8}$$

Using (6) and (8), we obtain from (5),

$$n(r)=\frac{m^3c^3}{3\pi^2\hbar^3}q^3\left[1+\frac{\pi^2}{(\beta mc^2)^2}\frac{2q^2+1}{2q^4}+\cdots\right] . \tag{9}$$

Using (9), Poisson's equation then becomes

$$\begin{aligned}\nabla^2\varphi&=4\pi e n\\ &\approx 4\pi e\frac{m^3c^3}{3\pi^2\hbar^3}\left[\frac{(\mu+mc^2+e\varphi)^2}{m^2c^4}-1\right]^{3/2}\\ &\quad\times\left[1+\frac{\pi^2}{2(\beta mc^2)^2}\left[\frac{(\mu+mc^2+e\varphi)^2}{m^2c^4}-1\right]^{-2}\right] .\end{aligned} \tag{10}$$

Putting

$$\begin{gathered}\phi=\frac{\varphi+\mu/e}{Ze/r}, \quad r=ax, \quad a\equiv\left[\frac{9\pi^2}{128Z}\right]^{1/3}\frac{\hbar^2}{me^2}\\ \lambda\equiv\left[\frac{4Z^2}{3\pi}\right]^{2/3}\frac{e^4}{\hbar^2c^2}, \quad v\equiv\frac{\pi^2}{32}\left[\frac{3\pi}{4Z^2}\right]^{4/3}\frac{\hbar^4}{\beta^2m^2e^8} ,\end{gathered} \tag{11}$$

Eq. (10) becomes

$$\frac{d^2\phi}{dx^2}=\frac{\phi^{3/2}}{x^{1/2}}\left[1+\lambda\frac{\phi}{x}\right]^{3/2}\left[1+v\frac{x^2}{\phi^2}\left[1+\lambda\frac{\phi}{x}\right]^{-2}\right] . \tag{12}$$

Equation (12) is a generalization of the Thomas-Fermi equation to include both the relativistic and thermal effects when the thermal effects are weak. Observe that (i) in the absence of relativistic effects, Eq. (12) reduces to the Marshak-Bethe [4] equation,

$$\frac{d^2\phi}{dx^2}=\frac{\phi^{3/2}}{x^{1/2}}\left[1+v\frac{x^2}{\phi^2}\right] ; \tag{13}$$

(ii) in the absence of thermal effects, Eq. (12) reduces to the Vallarta-Rosen [3] equation,

$$\frac{d^2\phi}{dx^2}=\frac{\phi^{3/2}}{x^{1/2}}\left[1+\lambda\frac{\phi}{x}\right]^{3/2} . \tag{14}$$

B. The total kinetic energy

The total kinetic energy of the electrons in the atom is given by

$$E_{\text{kin}}=\int_0^\infty 4\pi r^2dr\frac{1}{\pi^2\hbar^3}\int_0^\infty\frac{(\sqrt{p^2c^2+m^2c^4}-mc^2)p^2dp}{\exp\left[\frac{\sqrt{p^2c^2+m^2c^4}-mc^2-e(\varphi-\varphi_0)}{KT}\right]+1} . \tag{15}$$

Making Juttner's [12] transformation (2), (15) becomes

$$E_{\text{kin}}=\int_0^\infty 4\pi r^2dr\frac{m^4c^5}{\pi^2\hbar^3}\int_0^\infty\frac{(\sinh^2\theta\cosh^2\theta-\sinh^2\theta\cosh\theta)d\theta}{\frac{1}{\Lambda}e^{\beta mc^2\cosh\theta}+1} . \tag{16}$$

Putting

$$u=\beta mc^2\cosh\theta, \quad \frac{d\Psi(u)}{du}=\sinh\theta\cosh^2\theta-\sinh\theta\cosh\theta , \tag{17}$$

(16) becomes

$$E_{\text{kin}}=\int_0^\infty 4\pi r^2dr\frac{m^4c^5}{\pi^2\hbar^3}\frac{1}{\beta mc^2}\int_0^\infty\frac{d\Psi(u)/du}{\frac{1}{\Lambda}e^u+1}du . \tag{18}$$

Using Sommerfeld's [14] Lemma (6) again, (18) becomes

$$E_{\mathrm{kin}}=\int_0^\infty 4\pi r^2 dr\frac{m^4c^5}{\pi^2\hbar^3}\frac{1}{\beta mc^2}\times\left[\Psi(u_0)+\frac{\pi^2}{6}\Psi''(u_0)+\cdots\right]. \quad (19)$$

Introducing q as in (7), we obtain

$$\Psi(u_0)=\beta mc^2\left[\frac{q(q^2+1)^{3/2}}{4}-\frac{q\sqrt{q^2+1}}{8}-\tfrac{1}{8}\ln(q+\sqrt{q^2+1})-\frac{q^3}{3}\right],$$
$$\Psi''(u_0)=\frac{1}{\beta mc^2}\left[\frac{(1+3q^2)\sqrt{q^2+1}}{q}-\frac{2q^2+1}{q}\right]. \quad (20)$$

Using (20), and considering the relativistic effects to be weak (so that $x\ll 1$), (19) becomes

$$E_{\mathrm{kin}}\approx\int_0^\infty 4\pi r^2 dr\frac{m^4c^5}{8\pi^2\hbar^3}q^5\left[1+\frac{2\pi^2}{(\beta mc^2)^2}\frac{1}{q^4}+\cdots\right]. \quad (21)$$

Using (7), and rescaling φ and r, as in (11), (21) becomes

$$E_{\mathrm{kin}}=\frac{3}{5}\frac{Z^2e^2}{a}\left[\int_0^\infty dx\,x^2\left[\frac{\phi}{x}+\lambda\frac{\phi^2}{x^2}\right]^{5/2}+\tfrac{5}{8}v\int_0^\infty dx\,x^2\left[\frac{\phi}{x}+\lambda\frac{\phi^2}{x^2}\right]^{1/2}\right], \quad (22)$$

where ϕ satisfies Eq. (12). In order to evaluate (22) analytically, we will follow the procedure given by Schwinger [15] for the case without thermal effects, i.e., $v=0$.

Noting that $\lambda\ll 1$ and $v\ll 1$ (since the relativistic and thermal effects have been taken to be weak), (22) and (12) become

$$E_{\mathrm{kin}}=\frac{3}{5}\frac{Z^2e^2}{a}\left[\int_0^\infty dx\frac{\phi^{5/2}}{x^{1/2}}+\tfrac{5}{2}\lambda\int_0^\infty dx\frac{\phi^{7/2}}{x^{3/2}}+\tfrac{5}{16}\lambda v\int_0^\infty dx\,x^{1/2}\phi^{3/2}+\tfrac{5}{8}v\int_0^\infty dx\,x^{3/2}\phi^{1/2}\right], \quad (23)$$

and

$$\frac{d^2\phi}{dx^2}=\frac{\phi^{3/2}}{x^{1/2}}+\frac{3}{2}\lambda\frac{\phi^{5/2}}{x^{3/2}}-\frac{1}{2}\lambda v x^{1/2}\phi^{1/2}+v\frac{x^{3/2}}{\phi^{1/2}}. \quad (24)$$

Upon integrating by parts, and using Eq. (24), (23) becomes

$$E_{\mathrm{kin}}=\frac{3}{7}\frac{Z^2e^2}{a}\left[-\phi'(0)+\lambda\left[2\int_0^\infty dx\frac{\phi^{7/2}}{x^{3/2}}+3\int_0^\infty dx\frac{\phi^{5/2}\phi'}{x^{1/2}}+\frac{31}{16}v\int_0^\infty dx\,x^{1/2}\phi^{3/2}\right]-\frac{81}{8}v\int_0^\infty dx\,x^{3/2}\phi^{1/2}\right]. \quad (25)$$

Equation (25) shows that (i) the relativistic effects ($\lambda\neq 0$) enhance the total kinetic energy[1] [which is due to the enhanced concentration of the electrons near the nucleus (Muller and Rafelski [16] and Ferreirinho, Ruffini, and Stella [17])]—This is also in accord with the numerical calculation of Hill, Grout, and March [18]; (ii) the thermal effects reduce the total kinetic energy (which is due to the thermal expansion of the semiclassical radius of an atom)—this is in accord with the numerical calculations of Feynman, Metropolis, and Teller [5].

It is to be noted that a straightforward integration of the relativistic integrals in (25) fails because most of the relativistic correction originates in the region close to the nucleus where the Thomas-Fermi model breaks down. This difficulty may be remedied by taking the nucleus to have a finite size in the model (Hill, Grout, and March [18]). Alternately, one may seek to treat the strongly bound electrons near the nucleus correctly by using a separate procedure (Englert [19]).

III. GENERALIZED THOMAS-FERMI THEORY FOR AN ATOM IN A VERY STRONG MAGNETIC FIELD

A. Equation for the self-consistent potential

For an atom in a very strong magnetic field, the magnetic confinement of the electrons perpendicular to the magnetic field will dominate the Coulomb attractions by the nucleus. Thus, the electrons tend to move in thin cylindrical shells with the axes directed along the magnetic field, and precess around the nucleus.

For very strong magnetic fields, the Coulomb forces do not excite the electrons from the ground state, which is taken to comprise eigenstates of the angular momentum along the magnetic field, with spins antiparallel to the field, and zero excitation of the radial motion. The oscillatory motion of the electrons along the magnetic field is confined by the Coulomb field.

Consider a heavy atom with an atomic number Z and

[1]Schwinger [15] also sought to derive the formula (25), in the cold electron-gas limit $v\to 0$, but missed the first term (which is the dominating one and has the right sign) in the relativistic correction; the other term is subdominant and has the wrong sign.

let $\phi(\mathbf{r})$ be the Coulomb potential due to the nucleus and the distribution of electronic charge around it. In order to calculate the number density of the electrons in the ground state, we count the number of states inside a small volume taken to be a thin cylindrical shell of radius ρ, thickness $\Delta\rho$, and height Δz. The number of transverse states (which correspond to cyclotron orbits) is given by (Banerjee, Constantinescu, and Rehak [20]),

$$\Delta N_{\|} = \left[\frac{eB}{\hbar c}\right]\rho\Delta\rho \ . \tag{26}$$

The number of longitudinal states inside Δz is given by

$$\Delta N_{\|} = \frac{2\Delta z}{h}\int_0^\infty \frac{dp}{\exp\left[\dfrac{\sqrt{p^2c^2+m^2c^4}-mc^2-e(\phi-\phi_0)}{\kappa T}\right]+1} , \tag{27}$$

where

$$\phi_0 \equiv -\frac{\mu}{e}$$

and μ is the chemical potential of this electronic system. (We are assuming the electrons behave in a nondegenerate manner with respect to the slow longitudinal motions.)

The number of electrons in the cylindrical shell in question is then

$$\Delta N = \Delta N_{\perp}\Delta N_{\|} \tag{28}$$

and the number density of the electrons is given by

$$n(\mathbf{r}) = \frac{\Delta N}{2\pi\rho\Delta\rho\Delta z} \ . \tag{29}$$

On using (26) and (27), we then obtain

$$n(\mathbf{r}) = \frac{eB}{2\pi^2\hbar^2 c}\int_0^\infty \frac{dp}{\exp\left[\dfrac{\sqrt{p^2c^2+m^2c^4}-mc^2-e(\phi-\phi_0)}{\kappa T}\right]+1} \ . \tag{30}$$

Making Juttner's [12] transformation,

$$\frac{p}{mc} = \sinh\theta \ , \tag{31}$$

(30) becomes

$$n(\mathbf{r}) = \frac{eBm}{2\pi^2\hbar^2}\int_0^\infty \frac{\cosh\theta d\theta}{\dfrac{1}{\Lambda}e^{\beta mc^2\cosh\theta}+1} , \tag{32}$$

where

$$\beta = \frac{1}{\kappa T}, \quad \frac{1}{\Lambda} = e^{-\beta[\mu+mc^2+e\phi(\mathbf{r})]} \ .$$

Let us put

$$u = \beta mc^2\cosh\theta, \quad \frac{d\Psi(u)}{du} = \coth\theta \ , \tag{33}$$

so that (32) can be written as

$$n(\mathbf{r}) = \frac{eBm}{2\pi^2\hbar^2}\frac{1}{\beta mc^2}\int_0^\infty \frac{d\Psi(u)/du}{\dfrac{1}{\Lambda}e^u+1}du \ . \tag{34}$$

The integral in (34) can be evaluated by using Sommerfeld's [14] Lemma (6),

$$n(\mathbf{r}) = \frac{eBm}{2\pi^2\hbar^2}\frac{1}{\beta mc^2}\left[\Psi(u_0)+\frac{\pi^2}{6}\Psi''(u_0)+\cdots\right] . \tag{35}$$

Introducing q as in (7), we obtain

$$\Psi(u_0) = (\beta mc^2)q, \quad \Psi''(u_0) = -\left[\frac{1}{\beta mc^2}\right]\frac{1}{q^3} \ . \tag{36}$$

Using (36), (35) becomes

$$n(\mathbf{r}) = \frac{eBm}{2\pi^2\hbar^2}q\left[1-\frac{\pi^2}{(\beta mc^2)^2}\frac{1}{6q^4}+\cdots\right] . \tag{37}$$

Using (37), Poisson's equation then becomes

$$\nabla^2\phi = 4\pi en = 4\pi e\frac{eBm}{2\pi^2\hbar^2}\left[\frac{(\mu+mc^2+e\phi)^2}{m^2c^4}-1\right]^{1/2} \times\left[1-\frac{\pi^2}{6(\beta mc^2)^2}\times\left[\frac{(\mu+mc^2+e\phi)^2}{m^2c^4}-1\right]^{-2}\right] . \tag{38}$$

Putting

$$\phi+\frac{\mu}{e} = \frac{Ze}{r}\Phi, \quad r = \tilde{b}x$$
$$\tilde{b} = 2^{-3/5}\pi^{2/5}\alpha^{4/5}a_0 Z^{1/5}(B/\tilde{B})^{-2/5} , \tag{39}$$
$$\tilde{B} = \frac{m^2c^3}{\hbar e}, \quad \alpha \equiv \frac{e^2}{\hbar c}, \quad a_0 \equiv \frac{\hbar^2}{me^2} ,$$

and assuming the ground state of the atom to be spherically symmetric, even in the presence of a strong magnetic field (which is valid if the cylindrical shells in which the electrons move are distributed uniformly in a large sphere), Eq. (38) becomes

$$\frac{d^2\Phi}{dx^2}=(x\Phi)^{1/2}\left[1+\tilde{\lambda}\frac{\Phi}{x}\right]^{1/2}\left[1-\tilde{v}\frac{x^2}{\Phi^2}\left[1+\tilde{\lambda}\frac{\Phi}{x}\right]^{-2}\right], \tag{40}$$

where

$$\tilde{\lambda}=\frac{Ze^2}{2mc^2\tilde{b}},\quad \tilde{v}=\frac{\pi^2}{24}\left[\frac{\kappa T}{Ze^2/\tilde{b}}\right]\ll 1 .$$

Equation (40) is a generalization of the Kadomtsev equation to include both the relativistic and thermal effects when the thermal effects are weak. Observe that

(i) in the absence of relativistic effects, Eq. (40) reduces to the equation derived by Shivamoggi and Schram [9]

$$\frac{d^2\Phi}{dx^2}=(x\Phi)^{1/2}\left[1-\tilde{v}\frac{x^2}{\Phi^2}\right]; \tag{41}$$

(ii) in the absence of thermal effects, Eq. (40) reduces to the equation derived by Hill, Grout, and March [7] and Shivamoggi and Mulser [8]:

$$\frac{d^2\Phi}{dx^2}=(x\Phi)^{1/2}\left[1+\tilde{\Lambda}\frac{\Phi}{x}\right]^{1/2}. \tag{42}$$

B. The total kinetic energy

The total kinetic energy of the electrons in the atom is given by

$$E_{\text{kin}}=\int_0^\infty 4\pi r^2 dr\frac{eB}{2\pi^2\hbar^2 c}\int_0^\infty\frac{(\sqrt{p^2c^2+m^2c^4}-mc^2)dp}{\exp\dfrac{[\sqrt{p^2c^2+m^2c^4}-mc^2-e(\phi-\phi_0)]}{\kappa T}+1}. \tag{43}$$

Making Juttner's [12] transformation (31), (43) becomes

$$E_{\text{kin}}=\int_0^\infty 4\pi r^2 dr\frac{eBm^2c^2}{2\pi^2\hbar^2}\int_0^\infty\frac{(\cosh^2\theta-\cosh\theta)d\theta}{\frac{1}{\Lambda}e^{\beta m^2\cosh\theta}+1}. \tag{44}$$

Putting

$$u=\beta mc^2\cosh\theta,\quad \frac{d\Psi(u)}{du}=\frac{\cosh^2\theta-\cosh\theta}{\sinh\theta}, \tag{45}$$

(44) becomes

$$E_{\text{kin}}=\int_0^\infty 4\pi r^2 dr\frac{eBm^2c^2}{2\pi^2\hbar^2}\frac{1}{\beta mc^2}\int_0^\infty\frac{d\Psi(u)/du}{\frac{1}{\Lambda}e^u+1}du . \tag{46}$$

Using Sommerfeld's [14] Lemma (6) again, (47) becomes

$$E_{\text{kin}}=\int_0^\infty 4\pi r^2 dr\frac{eBm^2c^2}{2\pi^2\hbar^2}\frac{1}{\beta mc^2}\times\left[\Psi(u_0)+\frac{\pi^2}{6}\Psi''(u_0)+\cdots\right]. \tag{47}$$

Introducing q as in (7), we obtain

$$\Psi(u_0)=\beta mc^2[\tfrac{1}{2}(\sinh^{-1}q+q\sqrt{1+q^2})-q],$$
$$\Psi''(u_0)=\frac{1}{\beta mc^2}\frac{1}{q^3}[(1+q^2)^{3/2}-1]. \tag{48}$$

Using (48), and considering the relativistic effects to be weak (so that $x\ll 1$), (47) becomes

$$E_{\text{kin}}=\frac{eBm^2c^2}{4\pi^2\hbar^2}\int_0^\infty\frac{q^3}{3}\left[1+\frac{\pi^2/2}{(\beta mc^2)^2}\frac{1}{q^6}+\cdots\right]4\pi r^2 dr . \tag{49}$$

Using (7), and rescaling ϕ and r as in (39), (49) becomes

$$E_{\text{kin}}=\frac{1}{3}\frac{Z^2e^2}{\tilde{b}}\int_0^\infty x^2 dx\left[\left[\frac{\Phi}{x}+\tilde{\lambda}\frac{\Phi^2}{x^2}\right]^{3/2}+3\tilde{v}\left[\frac{\Phi}{x}+\tilde{\lambda}\frac{\Phi^2}{x^2}\right]^{-1/2}\right], \tag{50}$$

where Φ satisfies Eq. (40).

In order to evaluate (50) analytically, we will follow the procedure given by Schwinger [15] for the atom without a magnetic field. Noting that $\tilde{\lambda}\ll 1$ and $\tilde{v}\ll 1$ (since the relativistic and thermal effects have been taken to be weak), (50) and (40) become

$$E_{\text{kin}}=\frac{1}{3}\frac{Z^2e^2}{\tilde{b}}\left[\int_0^\infty dx\,\Phi^{3/2}x^{1/2}+\frac{3\tilde{\lambda}}{2}\int_0^\infty dx\,\Phi^{5/2}x^{-1/2}+3\tilde{v}\int_0^\infty dx\,\Phi^{-1/2}x^{5/2}+3\tilde{\lambda}\tilde{v}\int_0^\infty dx\,\Phi^{1/2}x^{3/2}\right], \tag{51}$$

$$\frac{d^2\Phi}{dx^2}=x^{1/2}\Phi^{1/2}+\frac{\tilde{\lambda}}{2}\Phi^{3/2}x^{-1/2}-\tilde{v}\Phi^{-3/2}x^{5/2}+\frac{3}{2}\tilde{\lambda}\tilde{v}\Phi^{-1/2}x^{3/2}. \tag{52}$$

Upon integrating by parts, and using Eq. (52), (51) becomes

$$E_{\rm kin}=\frac{1}{3}\frac{Z^2e^2}{\tilde{b}}\left[-\frac{1}{3}\Phi'(0)+\frac{19}{15}\tilde{\lambda}\int_0^\infty dx\,\Phi^{5/2}x^{-1/2}-\frac{4}{3}\tilde{v}\int_0^\infty dx\,\Phi^{-1/2}x^{5/2}+\frac{15}{2}\tilde{\lambda}\tilde{v}\int_0^\infty dx\,\Phi^{1/2}x^{3/2}\right]. \tag{53}$$

In the absence of the thermal effects ($\tilde{v}=0$), (53) reduces to the formula derived by Shivamoggi and Mulser [8], while in the absence of relativistic effects($\tilde{\lambda}=0$), (53) reduces to the formula derived by Shivamoggi and Schram [9]. (53) shows that (i) the relativistic effects ($\tilde{\lambda}\neq 0$) enhance the total kinetic energy of the electrons (which is again due to the enhanced concentration of the electrons near the nucleus); (ii) the thermal effects ($\tilde{v}\neq 0$) reduce the total kinetic energy of the electrons (which is, of course, due to the thermal expansion of the semiclassical size of the atom).

IV. DISCUSSION

In the present paper, we have given unified relativistic formulations with thermal effects of the Thomas-Fermi model for an atom within and without a very strong magnetic field. We have given generalizations of the Thomas-Fermi equation for an atom and the Kadomtsev equation for an atom in a very strong magnetic field to include both the relativistic and the thermal effects when the latter are weak. We have also given analytic calculations of the total kinetic energy of the electrons in the atom.

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